The 2D Ising ferromagnet: spreading of damage and its conjugate field



1994 J. Phys. A: Math. Gen. 27 L809

(http://iopscience.iop.org/0305-4470/27/21/004)

LETTER TO THE EDITOR

# The 2D Ising ferromagnet: spreading of damage and its conjugate field

Francisco A Tamarit† and Ladário Da Silva‡

Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, RJ, Brazil

Received 17 August 1994

**Abstract.** We study, through numerical simulations, the spreading of damage on the bi-dimensional ferromagnetic Ising model submitted to a new kind of external field $h$ recently introduced in the literature in the context of cellular automata. The field $h$ is defined as the frequency at which different random numbers are used for updating the two replicas. We show that $h$ is the conjugate field to the Hamming distance, i.e. it *destroys* the dynamical continous transition observed at $h = 0$, and the associated susceptibility *diverges* at the critical temperature.

In the last few years, the technique of spreading of damage has been applied to a great variety of models, reflecting its increasing relevance. Although it was originally introduced to study sensitivity to initial conditions in complex systems, this technique has also proved to be a very useful tool in the study of magnetic systems [1–14]. It allows one to get valuable information about the phase-space structure of unsolved models, like short-range spin glasses [7–9]. On the other hand, it provides a powerful method for calculating numerically the magnetic susceptibility of Ising-like systems [11].

The main idea of this technique consists in measuring the time evolution of the Hamming distance (the damage)

$$D(t) = \frac{1}{4N} \sum_{i=1}^{N} (S_i(t) - S_i'(t))^2 \qquad S_i = \pm 1 \tag{1}$$

between two different configurations $\{S\}$ and $\{S'\}$, submitted to the *same* thermal noise, i.e. using the *same* sequence of random numbers for updating the spins in the Monte Carlo simulations. The dependence of the final Hamming distance on the temperature of the system and on any other relevant parameter, allows us to construct a *dynamical phase diagram* which usually depends on the specific stochastic process used to implement the simulation (heat bath, Metropolis, Glauber, generalized dynamics [15], etc). When the final Hamming distance is non-zero, we say that the system is in a *chaotic* phase, since it displays sensitivity to the initial conditions. When the final Hamming distance is zero, we say that the system is in a *frozen* phase. Numerical simulations of a great variety of magnetic models suggest a strong correlation between *dynamical* and *thermodynamical* phase transitions, but due to the lack of a general theory for the spreading of damage one cannot assume that they will necessarily agree.

† E-mail address: tamarit@cat.cbpf.br

‡ E-mail address: ladario@cat.cbpf.br

0305-4470/94/210809+04$19.50  

One of the most studied magnetic systems through this technique is the two-dimensional ferromagnetic Ising model with nearest-neighbour interactions. Since it possesses a non-trivial thermodynamical behaviour which has already been solved exactly and, on the other hand, can be simulated numerically on any conventional computer, it emerges as the ideal prototype to study correlations between the static and dynamical transitions. When the simulations are implemented with a heat bath stochastic process, the system displays a low-temperature *chaotic* phase (in which the final Hamming distance depends on the initial damage) and a high-temperature *non-chaoatic* phase. The corresponding critical temperature $T_d$ agrees with the critical temperature associated with the ferro-paramagnetic thermodynamical transition $k_B T_c/J \sim 2.269$ [7]. The transition is a continuous one and the corresponding order parameter is the long time asymptotic value of the Hamming distance (damage) $D$ between the two replicas of the system.

When an external magnetic field $B$ is applied to the system, we know that: (i) the thermodynamical phase transition is destroyed, i.e. the long time asympotic value of the order parameter (the magnetization) is non-zero for any finite temperature, and (ii) the zero-field magnetic susceptibility $\chi_m$ defined as

$$\chi_m \equiv \left.\frac{\partial M}{\partial B}\right|_{B=0} \tag{2}$$

diverges at the critical temperature $T_c$ where the continuous transition occurs. We then say that $B$ is the *conjugate field* of the order parameter. When the same external magnetic field is applied in the study of the damage spreading a different behaviour emerges, and the transition persists [3]. One can so conclude that $B$ is *not* the conjugate field associated with the Hamming distance.

In a recent work Tsallis and Martins [16] introduced a new kind of external field $h$, defined as *the frequency at which distinct random numbers are used to update the two replicas*. Their study was restricted to the phase diagram of the Domany–Kinzel cellular automaton. They defined a new susceptibility

$$\chi_d = \left.\frac{\partial D}{\partial h}\right|_{h=0} \tag{3}$$

associated with the damage $D$. They studied, through numerical simulations, the behaviour of both $D$ and $\chi_d$, and proved that conditions (i) and (ii), expressed above, are satisfied, that is, $h$ is the conjugate field associated with the order parameter $D$.

In this work we try to answer the question of whether this definition can be extended to magnetic systems, i.e. whether $h$ can be considered the conjugate field associated with the *chaotic–non-chaotic* dynamical transition observed in the two-dimensional ferromagnetic Ising model. We use the same definition introduced in [16] for the zero-field susceptibility associated to $D$ (equation (3)). We study the behaviour of the system when ruled by a sequential heat bath Monte Carlo process. At any time $t$, a site is chosen randomly and its spin updated accordingly to the following rule:

$$S_i(t+\mathrm{d}t) = \begin{cases} +1 & \text{with probability} \quad \left[1+\mathrm{e}^{-2\beta h_i}\right]^{-1} \\ -1 & \text{with probability} \quad \left[1+\mathrm{e}^{+2\beta h_i}\right]^{-1} \end{cases} \tag{4}$$

where $h_i$ is the local field at site $i$ at time $t$,

$$h_i = \sum_{j\neq i}^{N} J\, S_j \tag{5}$$

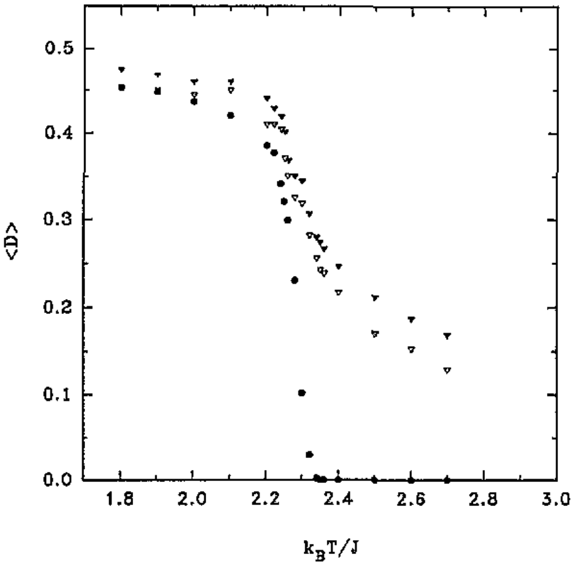


**Figure 1.** Spreading of damage $\langle D \rangle$ versus $k_B T/J$ for $N = 64$ and $h = 0$ (full circles), $h = 0.03$ (open triangles) and $h = 0.05$ (full triangles).

and $\beta \equiv (k_B T)^{-1}$. Since the updating is done sequentially, the natural scale for the Monte Carlo step is $dt = 1/N$. Starting from a previously thermalized configuration of the system $S_i(0)$, we make a copy $S_i'(0)$ of it in which each spin can be inverted with probability $\frac{1}{2}$ ($D(0) = \frac{1}{2} + O(1/\sqrt{N})$). Then, we let both configurations evolve, updating them with different random numbers with probability $h$ and with the same random numbers with probability $(1 - h)$. After a transient, we measure the time average of $D$ over a fixed number $\tau$ of Monte Carlo steps (MCS) which depends on the size of the system. At a given temperature, the same procedure is repeated $M$ times with different initial configurations and random sequences in order to get the mean value of the damage $\langle D \rangle$. In our simulations we worked with $L \times L$ systems of linear size 64, $\tau = 5000$ and $M = 1000$. Due to the lack of an equivalent to the fluctuation–dissipation theorem for the damage, we obtain the zero-field susceptibility by calculating the damage for several small values of $h$ (0.03, 0.05 and 0.10) and then deriving $\langle D \rangle$ with respect to $h$. Although this procedure may reduce the accuracy of the results, the main feature of the curve, that is, its divergent tendency at $T_d$ can be observed anyway.

In figure 1 we present the behaviour of $\langle D \rangle$ as a function of the temperature $T$ of the system, for $h = 0$, 0.03, and 0.05. Note that the field $h$ destroys the continous phase transition observed for $h = 0$. Figure 2 displays the zero-field susceptibility $\chi_d$ versus $T$ for different values of the field ($h = 0.03$ and 0.05). Note that it tends to diverge near the corresponding critical temperature as $h \to 0$, thus confirming that $h$ is the conjugate field associated to $D$. It is important here to stress that we concentrated our effort on observing the behaviour of the system when submitted to the field $h$, instead of trying to improve the zero-field result (via a finite-size effects study), since this last point has already been carefully studied [7].

Summarizing, in this work we have proved that the external field $h$, introduced by Tsallis and Martins [16] and defined as the frequency at which different random numbers are used in the updating of the two replicas, is the *conjugate field* associated to the Hamming

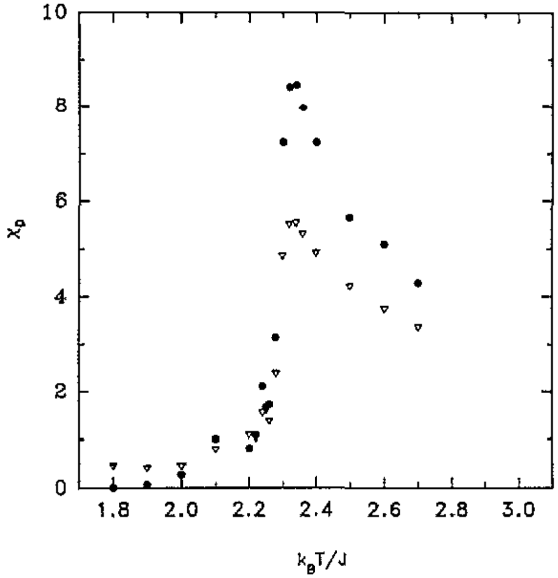


**Figure 2.** The susceptibility $\chi_d$ versus $k_BT/J$ for $N = 64$ and $h = 0.03$ (full circles) and 0.05 (open triangles).

distance. The present evidence is consistent with the mean-field results obtained in [17] for the Domany–Kinzel cellular automaton. It would be very interesting to obtain an equivalent version of the diffusion–dissipation theorem from which one could calculate with more accuracy, and more easily, the associated zero-field susceptibility. Although our study is restricted to the bi-dimensional ferromagnetic Ising model, we believe that this result can be extended to any transition observed through the technique of spreading of damage.

We acknowledge fruitful discussions with C Tsallis, M L Martins and D A Stariolo. We also thank CNPq (Brazil) for financial support. Part of the simulation was carried out on a Cray Y-MP2E of the Supercomputing Center at UFRGS.

## References

[1] Stanley H, Stauffer D, Kertés J and Herrmann H 1987 *Phys. Rev. Lett.* **59** 2326
[2] Costa U M S 1987 *J. Phys. A: Math. Gen.* **20** L583
[3] Le Caer G 1989 *J. Phys. A: Math. Gen.* **22** L647
[4] Leroyer Y and Rouidi K 1991 *J. Phys. A: Math. Gen.* **24** 1931
[5] Barber M and Derrida B 1988 *J. Stat. Phys.* **51** 877
[6] Neumann A and Derrida B 1988 *J. Physique* **49** 1647
[7] Derrida B and Weisbuch G 1987 *Europhys. Lett.* **4** 657
[8] da Cruz H, Costa U and Curado E 1989 *J. Phys. A: Math. Gen.* **22** L651
[9] Campbell I A and de Arcangelis L *Europhys. Lett.* **13** 587
[10] Martins M L, Verona H F, Tsallis C and de Magalhães A C N 1992 *Phys. Rev. Lett* **66** 2045
[11] Coniglio A, de Arcangelis L, Herrmann H and Jan N 1989 *Europhys. Lett.* **8** 315
[12] Mariz A 1989 *J. Phys. A: Math. Gen.* **23** 979
[13] Mariz A M, de Souza A M C and Tsallis C 1993 *J. Phys. A: Math. Gen.* **26** L1007
[14] Tamarit F and Curado E M F 1994 *J. Phys. A: Math. Gen.* **27** 671
[15] Mariz A M, Nobre F D and Tsallis C *Phys. Rev.* B **49** 3576
[16] Tsallis C and Martins M L 1994 *Europhys. Lett* at press
[17] Tomé T 1994 *Preprint*